Boson mappings and four-particle correlations in algebraic neutron-proton pairing models

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Abstract

Neutron-proton pairing correlations are studied within the context of two solvable models, one based on the algebra SO(5) and the other on the algebra SO(8). Boson-mapping techniques are applied to these models and shown to provide a convenient methodological tool both for solving such problems and for gaining useful insight into general features of pairing. We first focus on the SO(5) model, which involves generalized $T = 1$ pairing. Neither boson mean-field methods nor fermion-pair approximations are able to describe in detail neutron-proton pairing in this model. The analysis suggests, however, that the boson Hamiltonian obtained from a mapping of the fermion Hamiltonian contains a pairing force between bosons, pointing to the importance of boson-boson (or equivalently four-fermion) correlations with isospin $T = 0$ and spin $S = 0$. These correlations are investigated by carrying out a second boson mapping. Closed forms for the fermion wave functions are given in terms of the fermion-pair operators. Similar techniques are applied – albeit in less detail – to the SO(8) model, involving a competition between $T = 1$ and $T = 0$ pairing. Conclusions similar to those of the SO(5) analysis are reached regarding the importance of four-particle correlations in systems involving neutron-proton pairing.

21.60.-n, 21.60.Fw,21.60.Jz

I. INTRODUCTION

The residual interaction between the nucleons in a nucleus is expected to contain a strong neutron-proton pairing component on the basis of isospin-invariance arguments. Practical manifestations of neutron-proton pairing have proven elusive, however, in large part because most of the nuclei studied to date contain a significant neutron excess. In such nuclei, the neutrons and protons near the Fermi surface occupy different valence shells and thus cannot effectively exploit the neutron-proton pairing interaction. Moreover, as shown recently [1,2], even when the active neutrons and protons occupy the same valence shell, they cannot effectively build neutron-proton pair correlations except in the very narrow window of $N \approx Z$.

The development of radioactive beam facilities, now taking place at many laboratories worldwide, promises to change the experimental situation dramatically. With these new facilities, it should be possible to access all $N = Z$ nuclei up to ^{100}Sn. In many of the heavier $N \approx Z$ proton-rich nuclei, the neutron-proton pairing degree of freedom is expected to come into significant play. For this reason, renewed attention is now being devoted to the theoretical aspects of neutron-proton pairing. Since *full* shell-model calculations are feasible only for a limited set of nuclei, approximate methods are needed to study in detail such collective features. Historically, the method of choice has involved a generalization of the usual BCS treatment of pairing between like nucleons [3,4]. Unfortunately, this approach does not seem able to provide a correct description of many features of neutron-proton pair correlations [5]. Isospin projection seems to be a promising avenue to an improved theory [1], but up to now it has not been implemented.

In studying the effects of neutron-proton pairing, simple models can be very useful. Several such models, containing a semi-realistic representation of the different modes of pairing, have been constructed [6,7]. These models allow for exact solution and simple examination of various aspects of approximate treatments. Furthermore, they seem to reflect many of the key features of pairing that show up in more realistic shell-model calculations [1].

In the present paper, we employ the technique of boson mappings [8] to study pairing effects in these models. The basic idea of a boson mapping – as traditionally implemented – is to map bi-fermion operators onto boson operators in such a way as to preserve the physics of the original fermion problem. In principle, the original fermion problem can be completely solved within the boson space. In realistic applications, however, the boson mapping must be combined with approximation techniques.

Boson mappings not only provide a relatively simple methodology for treating the original fermion problem, but also can shed light on pairing correlations and on the applicability of pair approximations at the fermion level. The bosons are images of fermion pairs. Therefore, if the mapping does not result in a simple description in terms of the basis bosons, there would correspondingly not be a simple description in terms of fermion pairs.

Boson-mapping techniques may also be useful in an extended sense, by providing a natural methodology for incorporating correlations involving more than just two particles. In the neutron-proton pairing models to be discussed, inspection of the boson-mapped Hamiltonian suggests a pair collectivity between the bosons introduced in the mapping. This boson pair collectivity can then be treated with a second boson mapping [9], whereby bi-boson

operators are mapped onto new bosons representing quartets of the original fermions. The original fermion problem can then be rephrased in the language of these new bosons. A simple description in terms of these (quartet) bosons would confirm the importance of the associated four-fermion correlated structures.

The paper is organized as follows. We first consider the SO(5) model of monopole-isovector pairing. In Section II, we briefly discuss the model and its boson mapping and then examine in detail several variants of boson mean-field approximations. The applicability of fermion-pair approximations is then studied in Section III. In Section IV, a second boson mapping of the SO(5) model is performed and the role of bi-boson (four-fermion) structures is investigated. We then turn to the SO(8) model with both isoscalar and isovector degrees of freedom in Section V. The same issues are addressed as for the SO(5) model, but without as much detail. Finally, Section VI summarizes the key conclusions of the work and spells out some issues for future consideration.

II. SO(5) MODEL - BOSON MEAN-FIELD METHODS

A. The SO(5) model and its boson realization

The SO(5) model is perhaps the simplest tool to meaningfully investigate neutron-proton pairing. In this model, a system of N nucleons occupying a set of degenerate single-particle orbits with total degeneracy $4\Omega = 2\sum(2j+1)$ interact via an isovector pairing interaction,

$$H = gS^\dagger \cdot \widetilde{S} \ , \tag{1}$$

where

$$S^\dagger_\nu = \frac{1}{2}\sum_j \hat{\jmath}[a^{\dagger j\frac{1}{2}} a^{\dagger j\frac{1}{2}}]^{01}_{0\nu} \ \ , \tag{2}$$

and

$$\widetilde{S}_\nu = (-)^{1-\nu} S_{-\nu} \ . \tag{3}$$

The Hamiltonian (1) is invariant under the group of SO(5) tranformations generated by the three pair creation operators $S^\dagger_\nu$, the three conjugate pair annihilation operators S_ν, and the three components of the isospin operator $\mathcal{T}$. This makes the analysis of the model extremely simple and it is for this reason that it has been used recently in several studies of relevance to neutron-proton pairing [1,5,2].

A Dyson boson realization of the SO(5) algebra has been constructed in Ref. [10]. For the isovector pairing model, the boson space is constructed in terms of a scalar ($L=0$ $S=0$ $J=0$) isovector ($T=1$) boson s_ν, and the mapping takes the form [2]

$$\begin{aligned} S^\dagger_\nu &\to (\Omega - \hat{\mathcal{N}} + 1)s^\dagger_\nu - \frac{1}{2}s^\dagger \cdot s^\dagger \widetilde{s}_\nu \ , \\ \widetilde{S}_\nu &\to \widetilde{s}_\nu \ , \\ \mathcal{T}_\nu &\to \sqrt{2}[s^\dagger \widetilde{s}\,]^{01}_{0\nu} \ . \end{aligned} \tag{4}$$

Here, $\hat{\mathcal{N}} = -s^\dagger \cdot \widetilde{s}$ is the boson number operator. In these and all subsequent expressions, we use the standard notation for the scalar product,

$$s^\dagger \cdot s^\dagger = \sum (-)^\nu s^\dagger_\nu s^\dagger_{-\nu} \quad .$$

B. Approximate boson mean-field methods

Boson mappings provide an alternative technique for solving a fermion problem. Diagonalizing the mapped Hamiltonian in the ideal boson space (where possible) would yield all of the eigenvalues and the boson images of all eigenvectors of the original fermion problem. Note, however, that in the boson space spurious states may occur. These are boson states with no counterparts in the original fermion space that arise as a pure artifact of the mapping. In physically realistic problems, where it is impossible to diagonalize the mapped Hamiltonian in the full boson space, these spurious states cannot be readily removed from the problem. In the SO(5) model, however, this is not the case. Here it is possible to exactly diagonalize the mapped Hamiltonian. Furthermore, it can be shown that the spurious states only arise for $N > 2\Omega$ and that they all have isospin $T > 2\Omega - N/2$. Thus, by focusing our analysis on systems with $N \leq 2\Omega$, we can be sure that spurious boson states do not contaminate the physics of interest.

In the present study, we also treat the dynamics of the boson problem using approximate methods. By comparing with the exact results, we can assess the usefulness of these approximate methods in capturing the dominant collective dynamics in the presence of pair correlations.

We begin by considering boson mean-field techniques. The starting point is to introduce a mean-field boson γ as a linear combination of the basis bosons. The creation operator for the mean-field boson can thus be expressed as

$$\gamma^\dagger = \gamma_1 s^\dagger_1 + \gamma_{-1} s^\dagger_{-1} + \gamma_0 s^\dagger_0 \quad . \tag{5}$$

From this mean-field boson, several approximate variational states can be considered. Minimization of the energy of these variational states subject to the relevant constraints defines possible boson mean-field approximation procedures, each of which is the natural analog of a fermion variational procedure. We will denote each boson variational procedure by the corresponding standard fermion terminology.

In the boson analog of the BCS method, for example, the variational state is

$$|\mathrm{BSC}) \propto \exp(\eta\gamma^\dagger)|0) \ , \tag{6}$$

with constraints on the number of bosons (one-half the number of fermions)

$$(\mathrm{BCS}|\hat{N}/2|\mathrm{BCS}) = \mathcal{N}$$

and on the z-component T_z of the isospin

$$(\mathrm{BCS}|\mathcal{T}_0|\mathrm{BCS}) = T_z \quad .$$

The boson analog of number-projected BCS is the Hartree-Bose procedure, whereby the variational state is of the form of a boson condensate[1]

$$|\mathrm{BSC}, \mathcal{N}) \propto \gamma^{\dagger\mathcal{N}}|0) \tag{7}$$

with the single constraint

$$(\mathrm{BCS}, \mathcal{N}|\mathcal{T}_0|\mathrm{BCS}, \mathcal{N}) = T_z \quad .$$

The next level of mean-field approximation we consider involves the number-T_z projected BCS variational state,

$$|\mathrm{BSC}, \mathcal{N}\, T_z) \propto \mathcal{P}_{T_z}\gamma^{\dagger^{\mathcal{N}}}|0) \quad . \tag{8}$$

Here, $\mathcal{P}_{T_z}$ projects the boson condensate onto a state with definite T_z.

Finally, if we were to apply, in addition to number and T_z projection, full isospin T projection, the exact SO(5) state would be realized (as long as the probe function has a nonzero overlap with the exact state).

C. Energies

We now compare the energies arising at the various levels of boson mean-field approximation with the exact ground-state energies of the SO(5) Hamiltonian (1). We consider systems with an even number of nucleons only.

The exact eigenenergies for a system with $\mathcal{N}$ nucleon pairs are given by

$$E = -g[(\mathcal{N} - \frac{1}{2}v_s)(\Omega + \frac{3}{2} - \frac{1}{2}\mathcal{N} - \frac{1}{4}v_s) - \frac{1}{2}T(T+1)] \quad , \tag{9}$$

where v_s is the singlet-pairing seniority.

We are especially interested in the ground state of the system, which is realized for $v_s = 0$. The ground-state energy is then given by

$$\begin{aligned} E_{\mathrm{exact}} = -g[&\mathcal{N}(\Omega + \frac{3}{2} - \frac{1}{2}\mathcal{N}) \\ &-\frac{1}{2}T_z(T_z+1) - \delta(T_z+1)] \quad , \end{aligned} \tag{10}$$

where $\delta = 0$ for even-even ($T = T_z$) systems and 1 for odd-odd ($T = T_z + 1$) systems.

Applying the Dyson boson mapping (4) to the SO(5) Hamiltonian (1) leads to the boson Hamiltonian

$$H_{\mathrm{B}} = -g[\hat{\mathcal{N}}(\Omega + 1 - \hat{\mathcal{N}}) + \frac{1}{2}s^\dagger \cdot s^\dagger \widetilde{s} \cdot \widetilde{s}] \quad . \tag{11}$$

[1] Throughout the present paper, we understand the projection as performed before variation.

When this Hamiltonian is used in the variational approximations described in the preceding subsection, only two kinds of solutions can occur. The first, which we denote as A, has $\gamma_0 = 0$; the second, denoted by B, has $\gamma_1 = \gamma_{-1} = 0$, $\gamma_0 = 1$.[2]

In the BCS case, with trial state (6), the solutions A and B are degenerate for $T_z = 0$, whereas only solution A applies for $T_z > 0$. Note that this is precisely what was found in the generalized BCS treatment of Ref. [4]. The variational energy for these solutions is

$$E(\mathrm{BCS}) = E_{\mathrm{exact}} + g[\frac{3}{2}\mathcal{N} - \frac{1}{2}T_z - \delta(T_z+1)] \quad .$$

Adding number projection via the trial state (7) leaves many properties of the solution(s) unchanged. The solutions A and B are still degenerate for $T_z = 0$. And, for $T_z > 0$, only the solution A applies. There is a change in the variational energy, however, which now becomes

$$\begin{aligned} & E(\mathrm{BCS}, \mathcal{N}) \\ & \quad = E_{\mathrm{exact}} + g\mathcal{N}[1 - \frac{1}{2}\frac{T_z}{\mathcal{N}}(1 + \frac{T_z}{\mathcal{N}})] - g\delta(T_z+1) \quad . \end{aligned} \tag{12}$$

Note that for $T_z \rightarrow \mathcal{N}$, the approximate energy at this level of approximation approaches the exact result, as expected for a degenerate-orbit pairing model with one type of nucleon.

When imposing T_z projection as well, some new features appear. For even-even nuclei, for example, T_z projection removes the degeneracy of the solutions A and B, the solution A giving the lower energy. Thus, for even-even nuclei, the ground state solution is of the form A with variational energy

$$E(\mathrm{BCS}, \mathcal{N}\,T_z) = E_{\mathrm{exact}} + g\frac{1}{2}\mathcal{N}(1 - \frac{T_z}{\mathcal{N}}) \ . \tag{13}$$

Here, too, for $T_z \rightarrow \mathcal{N}$, the approximate result approaches the exact result, as clearly it should.

For odd-odd nuclei, on the other hand, the solution A disappears. Clearly, an odd-odd system must contain at least one $T_z = 0$ (s_0) boson, and such components are not present in solution A.

In fact, problems with the solution A in odd-odd nuclei already show up at the level of number-projected BCS approximation, though not as transparently. For an odd-odd nucleus with $T_z = \mathcal{N} - 1$, the number-projected variational energy (see Eq. 12) is lower than the exact energy.[3]

For the odd-odd $T_z = 0$ nucleus, the number-projected solution is thus of the form B and already has good T_z. Furthermore, the variational energy at this level of approximation is

$$E(\mathrm{BCS}, \mathcal{N}\,T_z = 0) = E_{\mathrm{exact}} + g(\mathcal{N} - 1) \quad .$$

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[2]This notation is in correspondence with Ref. [4].

[3]The same effect is seen in the fermion generalized BCS calculations of Ref. [4].

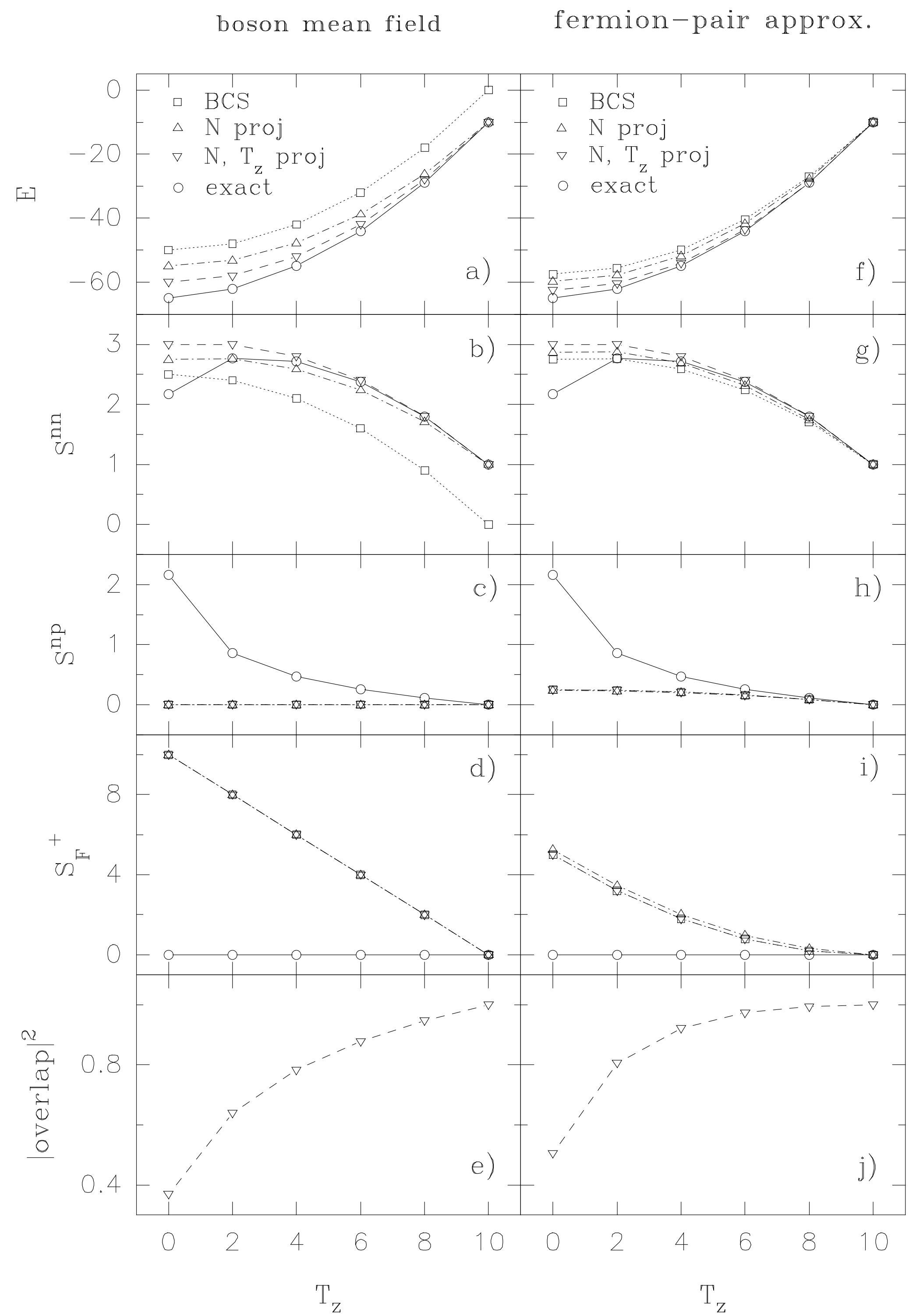

FIG. 1. The exact and BCS SO(5) ground-state energies E in units of g, two-particle transfer strengths $S^{\rm nn}$ and $S^{\rm np}$, the Fermi strengths $S_{\rm F}^{+}$, and overlaps of exact and BCS wave functions are shown as a function of T_z for $\Omega = 10$ and $\mathcal{N} = 10$. The results of the boson mean-field approximations and the fermion-pair approximations are displayed in the left and right parts of the figure, respectively. The exact results are denoted by circles, the BCS results by squares, the number projected BCS results by triangles pointing up, and the number-T_z projected BCS results by triangles pointing down.

For the odd-odd $T_z > 0$ nucleus, there is no number-T_z projected variational solution from the above-discussed class of trial functions.

A comparison of the variational energies obtained at the various levels of approximation discussed above with the exact ground-state energies is given in Fig. 1a. Two points are immediately evident from the figure. First, as expected, the successive restoration of symmetries improves the agreement with the exact results. Second, the variational energies, independent of the specific mean-field approximation, agree quite well with the exact values. This can be readily understood from the structure of the boson Hamiltonian (11). In this Hamiltonian, the first term only depends on $\hat{\mathcal{N}}$, and its contribution to the variational energy for states with a given number of bosons is not dependent on the form of the wave function. Moreover, the largest contribution to the variational energy from the second term $-\frac{1}{2}gs^\dagger \cdot s^\dagger \widetilde{s} \cdot \widetilde{s}$ in (11) is $-\frac{1}{2}g\mathcal{N}^2$ for all the above-discussed solutions. As a consequence, the ground-state energy is not a particularly sensitive observable for assessing the quality of the various approximate methods, at least for this model.

D. Two-particle transfer strengths

To assess the quality of the different variational solutions, we must consider other observables as well. In the SO(5) model, there are not many such possibilities. The pair operators multipled by $1/\sqrt{\Omega}$ give the normalized two-nucleon transfer strengths of the respective pair [11]. The mean values of the product of pair operators may then be connected with the summed strengths for two-nucleon pick-up from a given state i [12],

$$\begin{aligned} S^{\rm np} &= \langle \mathcal{N}, i | \frac{1}{\Omega} S_0^\dagger S_0 | \mathcal{N}, i \rangle \\ &= \sum_f | < \mathcal{N} - 1, f | \frac{1}{\sqrt{\Omega}} S_0 | \mathcal{N}, i \rangle |^2 \ , \end{aligned} \tag{14}$$

and similarly for $S^{\rm nn}$ and $S^{\rm pp}$.[4] Since two-neutron-transfer reactions provide valuable information on nn pairing correlations [13], it is expected that np transfer will analogously be sensitive to the neutron-proton-pairing mode. The sum of the three total transfer strengths is simply related to the energy of the spin-isospin conserving Hamiltonian (1).

Exact values of the total transfer strengths for each mode can be readily deduced from formulae given in Refs. [1,2]. For the approximate boson mean-field methods, we find

- BCS:

$$S^{\rm nn}(\text{solution A}) = \frac{1}{2}(\mathcal{N} + T_z)(1 - \frac{\mathcal{N} + T_z}{2\Omega}) \ ,$$

$$S^{\rm np}(\text{solution B}) = \mathcal{N}(1 - \frac{\mathcal{N}}{2\Omega}) \ .$$

[4]These mean values are related to the pairing gaps of the respective pairing modes [5]. Their usefulness as a rough measure of the number of the respective pairs has also been extensively discussed [1,4,2].

- Number-projected BCS:

$$S^{\rm nn}(\text{solution A}) = \frac{1}{2}(\mathcal{N}+T_z)(1-\frac{\mathcal{N}-1}{\mathcal{N}}\frac{\mathcal{N}+T_z}{2\Omega}) \ \ ,$$

$$S^{\rm np}(\text{solution B}) = \mathcal{N}(1-\frac{\mathcal{N}-1}{2\Omega}) \ \ .$$

- Number-T_z projected BCS:

$$S^{\rm nn}(\text{solution A}) = \frac{1}{2}(\mathcal{N}+T_z)(1-\frac{\mathcal{N}+T_z-2}{2\Omega}) \ \ ,$$

$$S^{\rm np}(\text{solution B}) = \mathcal{N}(1-\frac{\mathcal{N}-1}{2\Omega}) \ \ .$$

- All methods:

$$S^{\rm nn}(\text{solution B}) = 0 \ \ ,$$

$$S^{\rm np}(\text{solutionA}) = 0 \ \ .$$

The exact and aproximate results for the pair-transfer strengths are compared in Fig. 1b-c. The exact and approximate values differ considerably, much more so than the energies. For example, for $T = 0$ states, all three exact total transfer strengths are equal [1]. The approximate methods, however, give a zero transfer strength for the mode that is not present in the approximate wave function. Thus, even though the sum of all transfer strengths – as mirrored in the energy – is quite close to the exact value, its composition from the various terms of the Hamiltonian may be completely wrong.

E. Fermi strengths

Two other physically-interesting observables in the SO(5) model are the Fermi total strengths

$$S_{\rm F}^{+} = < \mathcal{T}^{-}\mathcal{T}^{+} > \ \ ,$$

$$S_{\rm F}^{-} = < \mathcal{T}^{+}\mathcal{T}^{-} > \ \ .$$

These two quantities are related by the Ikeda sum rule

$$S_{\rm F}^{-} - S_{\rm F}^{+} = 2T_z \ \ .$$

The exact expression for the $S_{\rm F}^{+}$ strength is

$$S^+_{\rm F}(\text{exact}) = 2\delta(T_z+1) \quad .$$

The results for the two types of approximate solutions, A and B, are independent of the specific boson mean-field method. They are given by

$$S^+_{\rm F}(\text{solution A}) = \mathcal{N} - T_z \quad ,$$

$$S^+_{\rm F}(\text{solution B}) = 2\mathcal{N} \quad .$$

A comparison of exact and approximate results is given in Fig. 1d. We see that the approximate results may differ substantially from the exact ones, especially for small values of T_z. This again suggests that even if the approximate wave functions provide reasonable values for energies their quality may be quite poor for other more-sensitive observables.

F. Overlaps

Another measure of the quality of approximate methods is the overlap of the approximate wave functions with the exact ones. Such a measure is not particularly useful when considering the BCS and number-projected BCS approximations, however, where the approximate wave functions are averaged over nuclei with different number of nucleon pairs and/or different T_z. Therefore, we calculate the overlaps only for the number-T_z projected BCS method. The results are illustrated in Fig. 1e.

We see that for small values of T_z, the approximate solution has very little overlap with the exact solution. It is only for $T_z \to \mathcal{N}$ that the overlap approaches one. In this limit, the solution A (a pure nn condensate) is the exact ground state of the model system. In the limit of symmetric nuclei, however, the approximate ground state wave function is very bad and a more sophisticated procedure is necessary in order to get an acceptable description of the system.

III. SO(5) MODEL - FERMION-PAIR APPROXIMATIONS

In this section, we discuss the solution of the SO(5) model in the original fermion space. While the exact solutions and the BCS solutions have been discussed before, the effect of number and T_z projection in the BCS approach has to our knowledge never been studied.

As noted in Subsect. II B, each of the boson mean-field approximations that we consider is the analogue of a fermion-pair approximation. The variational wave functions associated with these fermion-pair approximations can be readily obtained from (6), (7), and (8) by replacing the collective mean-field boson creation operator $\gamma^\dagger$ with a collective fermion-pair creation operator

$$\Gamma^\dagger = \Gamma_1 S^\dagger_1 + \Gamma_{-1} S^\dagger_{-1} + \Gamma_0 S^\dagger_0 \tag{15}$$

and the boson vacuum $|0)$ with the fermion vacuum $|0\rangle$.

Despite the obvious similarities between the boson mean-field and the corresponding fermion-pair approximations, they are not identical. Pauli effects are accomodated very

differently in the two approaches and this can lead to differences in the results. It is only in the limit $\Omega \to \infty$ that the corresponding bosonic and fermionic results agree. Nevertheless, we expect the main features found in the boson mean-field analysis to be present in the fermion-pair methods as well.

Such a conjecture is readily confirmed for the SO(5) model under investigation. In the fermion-pair approximations, for example, two types of solution likewise occur. There is a solution A corresponding to the $\mathrm{n\bar{n}} - \mathrm{p\bar{p}}$ phase ($\Gamma_0 = 0$) and a solution B corresponding to the $\mathrm{n\bar{p}} - \mathrm{p\bar{n}}$ phase ($\Gamma_1 = \Gamma_{-1} = 0$). Furthermore, as in the boson mean-field treatment, (*i*) T_z projection removes the degeneracy of the solutions A and B, and (*ii*) both solutions are unphysical for odd-odd nuclei, except in the case of $T_z = 0$ where the solution B is relevant.

Simple analytic expressions can be obtained in the fermion-pair approximations for many of the quantities discussed in Sect. II. In the fermion BCS, for example, the following simple expressions obtain:

$$E(\mathrm{BCS}) = E_{\mathrm{exact}} + g[\mathcal{N}(\frac{3}{2} - \frac{3\mathcal{N}}{4\Omega}) - \frac{1}{2}T_z(1 + \frac{T_z}{2\Omega}) - \delta(T_z + 1)] \ ,$$

$$S^{\mathrm{nn}}(\text{solution A}) = \frac{1}{2}(\mathcal{N} + T_z)(1 - \frac{\mathcal{N} + T_z}{2\Omega} + \frac{\mathcal{N} + T_z}{2\Omega^2}) \ ,$$

$$S^{\mathrm{nn}}(\text{solution B}) = \frac{\mathcal{N}^2}{4\Omega^2} \ ,$$

$$S^{\mathrm{np}}(\text{solution A}) = \frac{\mathcal{N}^2 - T_z^2}{4\Omega^2} \ ,$$

$$S^{\mathrm{np}}(\text{solution B}) = \mathcal{N}(1 - \frac{\mathcal{N}}{2\Omega} + \frac{\mathcal{N}}{4\Omega^2}) \ ,$$

$$S_{\mathrm{F}}^{+}(\text{solution A}) = \mathcal{N} - T_z - \frac{\mathcal{N}^2 - T_z^2}{2\Omega} \ ,$$

$$S_{\mathrm{F}}^{+}(\text{solution B}) = 2\mathcal{N}(1 - \frac{\mathcal{N}}{2\Omega}) \ .$$

For the number and number-T_z projected BCS, we have not obtained closed expressions for all of the quantities of interest, even though it is likely that they too can be derived. For those cases, the results we present have been obtained numerically. The one quantity for which we have obtained an analytic expression is the energy of the number-T_z projected state,

$$E(BCS, \mathcal{N}T_z) \ = \ E_{\mathrm{exact}} + \frac{g}{2}\mathcal{N}(1 - \frac{T_z}{\mathcal{N}} - \frac{\mathcal{N}^2 - T_z^2}{2\mathcal{N}\Omega}) \ . \tag{16}$$

In the right half of Fig.1, the results of the approximate fermion-pair methods are shown next to their corresponding boson mean-field results. The fermionic results are in general closer to the exact ones than their bosonic counterparts. Nevertheless, the main deficiencies found in the bosonic analysis persist when working directly in the fermion space. Most notable are the large disrepancies in the Fermi and two-particle transfer strengths and in the overlaps between the approximate and exact wave functions.

Closer inspection of the results indicates that even for the relatively well-reproduced ground-state energies the approximate methods do not capture some important details. An example is the double binding energy difference [14],

$$\delta V_{\mathrm{np}}(N,Z) = \frac{1}{4}\{[B(N,Z) - B(N-2,Z)] \\ -[B(N,Z-2) - B(N-2,Z-2)]\} \ .$$

For even-even nuclei in this model, the exact expression for this quantity is

$$\begin{aligned} \delta V_{\mathrm{np}}(N,Z) &= -\frac{1}{4}g \quad , \text{ for } N=Z \\ &= \ 0 \qquad , \text{ otherwise } . \end{aligned}$$

The jump of $\delta V_{\mathrm{np}}(N,Z)$ at the $N=Z$ line, which is a persistent feature of experimental data [15], may be used to isolate the Wigner term in the binding energy [16].[5]

When $\delta V_{\mathrm{np}}(N,Z)$ is calculated using the fermion number-T_z projected BCS method (16) – the approximation that yields the best reproduction of the exact energies – one obtains [2]

$$\delta V_{\mathrm{np}}(N,Z) = -\frac{1}{4\Omega}\ g \ ,$$

irrespective of whether $N=Z$ or not. Clearly, the physics of the jump at $N=Z$ is rather subtle and its correct description needs more sophisticated approximate methods than BCS or any of its variants.

Apparently, the standard fermion-pair approximations do not allow for the coexistence of like-particle and neutron-proton pairs *in the case of an isospin-conserving Hamiltonian.* This results in a very small value of the two-nucleon transfer strength for the mode not present in the wave function. As shown recently [5], all three pairing modes can coexist, however, when the isospin symmetry of the Hamiltonian is violated. This led to a proposal to introduce an isospin-breaking Hamiltonian, chosen to reproduce at BCS level the various pairing gaps in symmetric nuclei, and to use it in BCS treatments of all nuclei.

We question such a prescription. One can rephrase the prescription of Ref. [5] and look for the parameters Γ_ν in the general pair creation operator (15) that in $T=0$ nuclei give appropriate two-nucleon transfer strengths (or pairing gaps) for all three modes. The condition within the SO(5) model is that all three strengths should be equal. Working in the number-T_z projected BCS approximation and focusing on the case $\Omega=10$, $\mathcal{N}=10$, we

[5]Of course, the actual Wigner term may not have its main contribution coming from isovector pairing, as it does in the SO(5) model.

find that this equality can be achieved by choosing $\Gamma_1^2 = \Gamma_{-1}^2 = 0.2593$, and $\Gamma_0^2 = 0.4815$. When we then calculate the ground-state energy and the Fermi strength with this same choice of structure coefficients (for the same $T = 0$ nucleus), we obtain the results -10.71 and 108.6, respectively. These are to be compared with the exact results of -65.0 and 0, respectively, calculated using the isospin-invariant Hamiltonian. Clearly, the proposed prescription cannot describe in a unified way the various observables of interest.

We can understand these results as follows. The fermion-pair state just described, namely the one that yields equal pair-transfer strengths for all three modes, is in fact very close to the exact state with *maximal* isospin $T_{\max} = \mathcal{N}$,

$$|BCS : \mathcal{N}\, T_{\max}\, T_z\rangle \propto \mathcal{P}_{T_z}(\frac{1}{2}S_1^\dagger + \frac{1}{2}S_{-1}^\dagger + \frac{1}{\sqrt{2}}S_0^\dagger)^{\mathcal{N}}|0\rangle \ . \tag{17}$$

This suggests that the isospin-breaking procedure proposed in Ref. [5] produces high-isospin admixtures in the ground state that are too large. And this makes questionable its usefulness in describing the ground state of an isospin-conserving Hamiltonian when $T_z \approx 0$.

It is useful to summarize here the principal findings up to this point in the analysis. It has been argued many times that the (generalized) BCS method cannot properly describe neutron-proton pairing in the SO(5) model [5]. The present results show that this conclusion does not change when number and T_z projection are switched on (with variation after projection). The deficiencies of these standard fermionic methods for treating pairing are clearly seen in the analysis. And, furthermore, they can be alternatively seen within the context of analogous mean-field boson methods applied following a boson mapping of the model.

IV. SO(5) MODEL - BEYOND FERMION-PAIR CORRELATIONS

Since fermion-pair approximations and the corresponding boson mean-field methods show deficiencies when applied to the ground state of the isovector neutron-proton pairing SO(5) model, we need to consider more sophisticated procedures.

A. BCS for boson Hamiltonian

The boson-mapped SO(5) Hamiltonian (11) contains, in addition to the dominant linear term, an attractive boson-boson pairing interaction. Two bosons only interact when in a $J = 0$ $T = 0$ scalar-isoscalar state. A natural first approach to consider for a system dominated by boson pairing is boson BCS approximation [17].

In this approach, a variational boson wave function of the form[6]

$$|\Phi) \propto \exp(\eta s_1^\dagger + \zeta s^\dagger \cdot s^\dagger)|0) \tag{18}$$

is considered. A generalized Bogolyubov transformation,

---

[6]We limit our discussion here to even-even systems.

$$\sigma_\nu^\dagger = u s_\nu^\dagger - v \widetilde{s}_\nu - x \delta_{1\nu} \ , \tag{19}$$

is then introduced, with the constraint

$$u^2 - v^2 = 1 \quad .$$

The state (18) is the vacuum for quasiboson operators (19) if the relations

$$\begin{aligned} 2u\zeta + v = 0 \ , \\ u\eta - x = 0 \end{aligned}$$

are satisfied.[7] Constraints on the average number of pairs $\mathcal{N}$ and the average value of the isospin T_z give

$$\begin{aligned} v^2 &= \frac{\mathcal{N} - T_z}{2T_z + 3} \ , \\ x^2 &= T_z \quad . \end{aligned}$$

The energy of the state (18) is then easily calculated. We present explicitly the result for $T_z = 0$ only, for which the energy is

$$E(\mathrm{BCS}) = E_{\mathrm{exact}} + \mathcal{N}(1 + \frac{1}{3}\mathcal{N}) \quad . \tag{20}$$

An illustrative comparison of the exact and approximate boson BCS energies is given in the upper part of Fig. 2. We see that the boson BCS theory cannot explain the energies satisfactorily.

Its failure can be traced to a very large dispersion in the pair number contained in the wave function (18). For the model under discussion, the dispersion is given by

$$(\Delta\mathcal{N})^2 = (\Phi|\hat{\mathcal{N}}^2|\Phi) - \mathcal{N}^2 = \frac{(\mathcal{N} - T_z)(\mathcal{N} + T_z + 3)(8T_z + 6)}{(2T_z + 3)^2} + T_z \ . \tag{21}$$

The results are displayed in the lower part of Fig. 2. As is clearly evident, the dispersion is quite large, especially for small values of T_z.

---

[7]In fact, the wave function (6) of Subsect. II B can be considered in an analogous way with $v = 0$. We denote the method in the present subsection as the boson BCS to distinguish it from the BCS (6) in the mean-field boson approximation. In the latter, the BCS bra state is a boson image of the fermion BCS state.

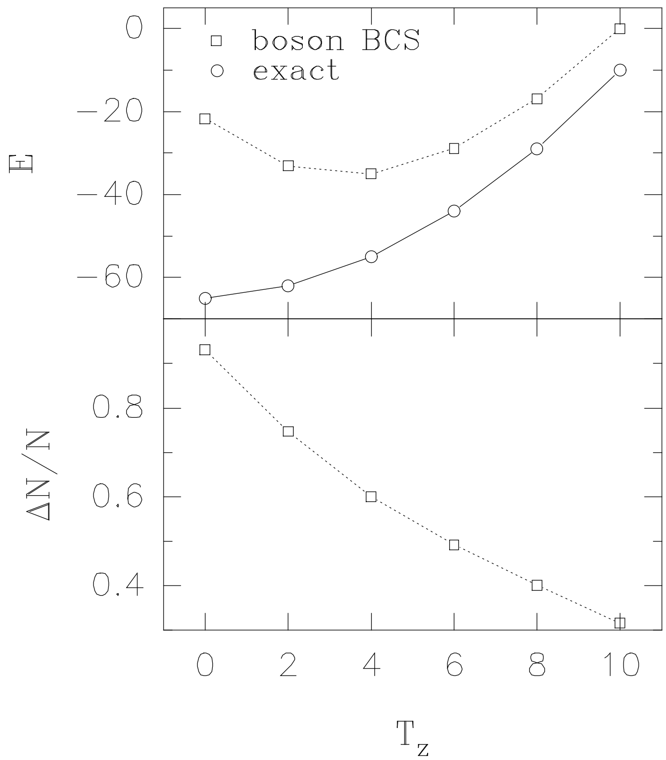

FIG. 2. The exact and boson BCS SO(5) ground-state energies in units of g are shown in the upper part of the figure as a function of T_z for $\Omega = 10$ and $\mathcal{N} = 10$. The pair-number dispersion in the boson BCS state is given in the lower part.

Some understanding of this result can be obtained by focussing on the case of T_z=0. Then, Eq.(21) is a special case of a formula for the dispersion associated with the BCS wave function (18) for $\eta = 0$ and for pairing in a single level of degeneracy 2Ω:

$$(\Delta\mathcal{N})^2 = \mathcal{N}^2/\Omega + 2\mathcal{N} \quad . \tag{22}$$

In the case under discussion of an isovector s boson, $2\Omega = 3$ and

$$(\Delta\mathcal{N})^2 = \frac{2}{3}\mathcal{N}(\mathcal{N} + 3) \quad . \tag{23}$$

Because of the inverse dependence on Ω of the dominant quadratic term in (22), the dispersion is quite large at T_z=0.[8]

Thus, due primarily to the small dimensionality of the s-boson space ($d = 3$), boson BCS is not an acceptable approach for treating boson-boson pairing correlations in this model. It should be noted, however, that when number and T_z projection are applied to (18) the exact ground-state results are obtained.

[8]Note that in boson BCS theory, as contrasted to fermion BCS theory, the linear and quadratic contributions add coherently, giving a further reason for the large dispersion.

B. A second boson mapping

As we saw in Section 2, a boson mapping followed by a mean-field treatment of the resulting Hamiltonian is an alternative to a BCS treatment of the original problem. Thus, we will now consider the possibility of implementing a second mapping of the problem, from the system of s_ν bosons (representing fermion pairs) to a new system of bosons that represent fermion quartets. We will first develop and apply the ideas to systems with $T=0$ and then subsequently (for reasons to be clarified later) to systems with $T \neq 0$.

1. $T=0$ case

In the boson-mapped SO(5) Hamiltonian (11) only the operators

$$s^\dagger \cdot s^\dagger \quad , \quad \widetilde{s} \cdot \widetilde{s} \quad , \quad s^\dagger \cdot \widetilde{s} \tag{24}$$

appear. This set of operators closes under commutation,

$$\begin{aligned}
[s^\dagger \cdot \widetilde{s}, s^\dagger \cdot s^\dagger] &= -2 s^\dagger \cdot s^\dagger \ , \\
[s^\dagger \cdot \widetilde{s}, \widetilde{s} \cdot \widetilde{s}] &= 2 \widetilde{s} \cdot \widetilde{s} \ , \\
[s^\dagger \cdot s^\dagger, \widetilde{s} \cdot \widetilde{s}] &= -4 s^\dagger \cdot \widetilde{s} + 6 \ \ ,
\end{aligned} \tag{25}$$

generating the algebra O(2,1).

In the second boson mapping, the boson space built up from the three s_ν bosons is mapped onto a new space built up in terms of a single $J=0$ $T=0$ boson, which we denote t. Whereas the s_ν bosons represented pairs of the original fermions, these new bosons represent pairs of s bosons, or equivalently quartets of the original fermions. It should be clear at this point that the mapping just outlined can only give information about states with total $T=0$.

In more detail, the mapping is constructed such that the commutation algebra of bi-boson operators (24) in the s-boson space are preserved in the t-boson space. The Dyson realization of the mapping for the O(2,1) algebra is

$$\begin{aligned}
s^\dagger \cdot s^\dagger &\rightarrow 6 t^\dagger + 4 t^\dagger t^\dagger t \ , \\
\widetilde{s} \cdot \widetilde{s} &\rightarrow t \quad , \\
s^\dagger \cdot \widetilde{s} &\rightarrow -2 t^\dagger t \ \ .
\end{aligned} \tag{26}$$

Here, $t^\dagger$ and t denote the creation and annihilation operators of the boson t.

We obtain after the second boson mapping a U(1) Hamiltonian

$$H_{\rm BB} = -2g(\Omega + \frac{3}{2} - t^\dagger t) t^\dagger t \ \ . \tag{27}$$

The eigenfunctions and eigenvalues of this Hamiltonian can be simply obtained. The eigenfunctions are condensates of t bosons, $|t^n))$ [9] and the corresponding eigenenergies are

[9] We use a double bracket)) to denote a state in the t-boson space.

$$E = -2g((\Omega + \frac{3}{2} - n)n \quad . \tag{28}$$

These eigenenergies are exactly equal to the ground-state energies (10) of the original fermion SO(5) Hamiltonian (1) if the obvious relation $n = \frac{1}{2}\mathcal{N}$ is invoked.

There are several points that should be made before proceeding to the $T \neq 0$ case:

- The exact eigenstate $|t^n))$ is in the form of a Hartree-Bose approximate solution for the Hamiltonian $H_{\rm BB}$. Equivalently, following the terminology of Sect. II B, we can think of it as representing the number-projected BCS approximation of $H_{\rm B}$. Thus, we have confirmed that BCS approximation of the mapped Hamiltonian $H_{\rm B}$ is an acceptable way to describe the full correlation structure of the problem, *as long as number projection is included.*

- Clearly, the solution $|t^n))$ involves four-fermion correlations since the boson t itself represents a correlated quartet. Thus, we have confirmed the essential role played by four-body correlations in systems involving both like-particle (nn and pp) pairing correlations and np pairing correlations.

- Lastly, we have demonstrated, through the example studied here, the possible usefulness of iterative boson mappings in accomodating many-particle correlation structures. Considering the difficulty in building quasiparticle methods that can treat many-particle (e.g., quartet) correlations directly at the original fermion level, we feel that this could be an important conclusion.

Of course, we should bear in mind that all of the above points have been demonstrated so far for $T = 0$ systems only.

2. $T \neq 0$ case

The boson space introduced in the previous subsection is constructed entirely in terms of the $J = 0$ $T = 0$ t boson. Thus, as noted earlier, the Hamiltonian (27) derived using the mapping (26) to that space can only provide information on even-$\mathcal{N}$ $J = 0$ $T = 0$ states.

To make possible a more complete analysis, we have to extend the set of operators (24) considered for the second boson mapping. If we want to also include nuclei with $\mathcal{N}$ = odd and/or $T_z \neq 0$, we have to add to the set of operators (24) the creation and annihilation operators of the s boson,

$$s^\dagger \quad , \quad \widetilde{s} \quad . \tag{29}$$

We should thus consider, in addition to (25), the commutation relations

$$\begin{aligned}
[\widetilde{s}_\nu, s^\dagger \cdot s^\dagger] &= -2s^\dagger_\nu \ , \\
[\widetilde{s}_\nu, \widetilde{s} \cdot \widetilde{s}] &= 0 \ , \\
[\widetilde{s}_\nu, s^\dagger \cdot \widetilde{s}] &= -\widetilde{s}_\nu \ ,
\end{aligned} \tag{30}$$

and their hermitian conjugates.

The algebra of the operators (24) and (29) have their Dyson boson realization on the space formed by the scalar-isoscalar boson t (representing, as before, fermion quartets) and the scalar-isovector boson σ_ν (representing fermion pairs).[10]

The Dyson realization of this extended algebra is

$$\begin{aligned} s^\dagger \cdot s^\dagger &\to 6t^\dagger + 4t^\dagger t^\dagger t + \sigma^\dagger \cdot \sigma^\dagger - 4t^\dagger \sigma^\dagger \cdot \widetilde{\sigma} \ , \\ \widetilde{s} \cdot \widetilde{s} &\to t \ \ , \\ s^\dagger \cdot \widetilde{s} &\to -2t^\dagger t + \sigma^\dagger \cdot \widetilde{\sigma} \ , \\ s^\dagger_\nu &\to \sigma^\dagger_\nu - 2t^\dagger \widetilde{\sigma}_\nu \ , \\ \widetilde{s}_\nu &\to \widetilde{\sigma}_\nu \ \ . \end{aligned} \tag{31}$$

The ideal space formed by the t and σ bosons is larger than the original s-boson space. Nonphysical (spurious) states are introduced by the second boson mapping, in addition to those already introduced by the first boson mapping. Full diagonalization of the mapped Hamiltonian in the ideal space separates physical states from spurious states, however, providing eigenvalues and wave function images of the original problem.

Applying the second boson mapping (31) to the boson Hamiltonian (11), we obtain

$$\begin{aligned} H_{\mathrm{BB}} = -g[\hat{\mathcal{N}}(\Omega + 1 - \hat{\mathcal{N}}) &+ 3t^\dagger t + 2t^\dagger t^\dagger tt \\ &-2t^\dagger t \sigma^\dagger \cdot \widetilde{\sigma} + \frac{1}{2}\sigma^\dagger \cdot \sigma^\dagger t] \ , \end{aligned} \tag{32}$$

where we have used the notation

$$\hat{\mathcal{N}} = 2t^\dagger t - \sigma^\dagger \cdot \widetilde{\sigma}$$

for the total number of nucleon pairs in the system. Note that the Hamiltonian (32), when restricted to states built from t bosons, correctly reduces to the $T = 0$ Hamiltonian (27).

Though the Hamiltonian (32) is nonhermitian (a general feature of Dyson mappings) with non-zero nondiagonal elements, we can nevertheless easily determine its eigenenergies. In the basis with good isospin T,

$$|t^{\frac{1}{2}(\mathcal{N}-n_\sigma)} \ ; \ \sigma^{n_\sigma} T)) \ \ , \ \ n_\sigma \geq T \ \ ,$$

the Hamiltonian matrix for (32) has all zero elements below the main diagonal. Therefore, the diagonal matrix elements coincide with its eigenvalues, which we find to be

$$-g[\mathcal{N}(\Omega + \frac{3}{2} - \frac{1}{2}\mathcal{N}) - \frac{1}{2}n_\sigma(n_\sigma + 1)] \ \ .$$

[10]This procedure is analogous to the familiar boson-fermion mappings. In those mappings, introduced to simultaneously describe systems with an even number and an odd number of fermions, the set of bi-fermion and fermion operators is mapped onto a space formed by boson and quasi-fermion operators.

The lowest eigenvalue, with $n_\sigma = T$, corresponds to the physical state.[11] The other eigenstates are nonphysical. The energies of the physical solutions are in precise agreement with the exact energies (9) of the $v_s = 0$ states, as they should be.

The corresponding left eigenvector for the physical $n_\sigma = T$ state is readily found to be

$$((t^{\frac{1}{2}(\mathcal{N}-T)}; \sigma^T T| \quad . \tag{33}$$

Note further that the exact left eigenstate is a product of a t-boson condensate and a σ-boson condensate, precisely the form of a coupled Hartree-Bose solution.

From the above remarks, we see that all of the conclusions given at the end of the preceding subsection (for $T = 0$ states) carry over to the more general case. Namely, (i) number-projected boson BCS approximation following the first boson mapping is an appropriate method for incorporating the correlations contained in the SO(5) model, (ii) incorporating four-particle correlations within the context of alpha-like clusters is the key to describing the structure of the system, and (iii) iterative boson mappings are an attractive means of accomodating the various correlations contained in the model.

C. Wave function in terms of fermion-pair operators

In the previous subsection, we obtained an analytic form for the left eigenstates of the SO(5) model in the $t - \sigma$ space of the second boson mapping. For this particular problem, we can invert the procedure to obtain the corresponding exact fermion wave functions.

The key to the procedure is to focus on the left eigenstates (33) of the second boson Hamiltonian. As can be seen from the mapping equations (4, 26 and 31), there is a simple chain of relations that take us from the left fermion eigenstates to the left second boson eigenstates, namely

$$\begin{aligned} \widetilde{S} \cdot \widetilde{S} &\rightarrow \widetilde{s} \cdot \widetilde{s} \rightarrow t \ , \\ \widetilde{S}_\nu \ &\rightarrow \widetilde{s}_\nu \rightarrow \widetilde{\sigma}_\nu \quad . \end{aligned} \tag{34}$$

We can therefore write the left fermion eigenstate that leads to (33) as[12]

$$\langle \mathcal{N} T| \propto \langle TT|(\widetilde{S} \cdot \widetilde{S})^{\frac{1}{2}(\mathcal{N}-T)} \quad .$$

Since at the fermion level the right and left eigenstates are conjugates to one another, we can now write the right eigenstate as

$$|\mathcal{N} T\rangle \propto (\widetilde{S} \cdot \widetilde{S})^{\frac{1}{2}(\mathcal{N}-T)} |TT\rangle \quad .$$

Finally, making use of the relation (17), we obtain for the exact SO(5) state vectors in terms of fermion pairs

---

[11] For $\mathcal{N} > \Omega$, the condition $T \leq 2\Omega - \mathcal{N}$ must also be taken into account.

[12] Note that the normalization of the fermion state cannot be simply deduced from the boson state, so that only a proportionality relation is given.

$$|\mathcal{N}TT_z\rangle$$
$$\propto \mathcal{P}_{T_z}(S^\dagger \cdot S^\dagger)^{\frac{1}{2}(\mathcal{N}-T)}(\frac{1}{2}S_1^\dagger + \frac{1}{2}S_{-1}^\dagger + \frac{1}{\sqrt{2}}S_0^\dagger)^T|0\rangle \ . \tag{35}$$

A special case of this relation for the ground states of even-even nuclei (with $T = T_z$) and odd-odd nuclei (with $T = T_z + 1$) was given in Ref. [2]. Note that the exact wave functions involve a condensate of $S^\dagger \cdot S^\dagger$ J=0 T=0 quartets and an isospin-stretched condensate of J=0 T=1 pairs projected to good T_z.

The above considerations reconfirm what was demonstrated in the previous subsections, namely that four-particle correlations are an essential ingredient for a correct description of the eigenstates of the neutron-proton pairing SO(5) model.

V. SO(8) MODEL

A. The model and its boson realization

A second algebraic model that has been used extensively [7,11,4] to study neutron-proton pairing correlations is one based on the algebra SO(8). As in the SO(5) model, neutrons and protons move in a set of degenerate single-particle orbits of total degeneracy 4Ω. In this model, however, they interact via a sum of an scalar-isovector pairing interaction *and* a vector-isoscalar pairing interaction, with the Hamiltonian taking the form[13]

$$H = \frac{g(1+x)}{2}\ S^\dagger \cdot \widetilde{S} + \frac{g(1-x)}{2}\ P^\dagger \cdot \widetilde{P} + g_{\rm ph}\ \mathcal{F} \cdot \mathcal{F} \ . \tag{36}$$

Here, the operator $S_\nu^\dagger$ creates the same L=0 S=0 J=0 T=1 isovector pair as in the SO(5) model, the operator $P_\nu^\dagger$ creates an L=0 S=1 J=1 T=0 isoscalar pair, and $\mathcal{F}_\nu^\mu$ is the Gamow-Teller operator.

Note that the relative importance of isoscalar and isovector pairing in the Hamiltonian (36) is governed by a single parameter x, which varies from -1 (pure isoscalar pairing) to $+1$ (pure isovector pairing). The last term in the Hamiltonian, a particle-hole force in the $T = 1$ $S = 1$ channel, is included to bring the Hamiltonian into closer contact with more realistic nuclear Hamiltonians, without destroying the simplicity of the model.

The Hamiltonian (36) is invariant under the group of SO(8) transformations generated by the three isovector pair creation operators $S_\nu^\dagger$, their three conjugate annihilation operators S_ν, the three isoscalar pair creation operators $P_\mu^\dagger$, their three conjugate annihilation operators P_μ, the three components of the isospin operator T_ν, the three components of the spin operator $\mathcal{S}_\mu$, and the nine components of the Gamow-Teller operator $\mathcal{F}_\nu^\mu$.

The Dyson boson realization of the SO(8) algebra is constructed by mapping its bi-fermion operators onto bosonic operators formed from the creation operators $s_\nu^\dagger$ of a scalar-isovector boson s and $p_\mu^\dagger$ of a vector-isoscalar boson p, and their conjugate annihilation operators $\widetilde{s}_\nu = (-)^{1-\nu}s_{-\nu}$ and $\widetilde{p}_\mu = (-)^{1-\mu}p_{-\mu}$ [18]:

---

[13]We only consider the isospin-conserving version of the SO(8) model in this work.

$$
\begin{aligned}
S^\dagger_\nu &\to (\Omega - \hat{\mathcal{N}} + 1)s^\dagger_\nu + \frac{1}{2}(p^\dagger \cdot p^\dagger - s^\dagger \cdot s^\dagger)\tilde{s}_\nu \ , \\
P^\dagger_\mu &\to (\Omega - \hat{\mathcal{N}} + 1)p^\dagger_\mu + \frac{1}{2}(s^\dagger \cdot s^\dagger - p^\dagger \cdot p^\dagger)\tilde{p}_\mu \ , \\
\tilde{S}_\nu &\to \tilde{s}_\nu \ , \\
\tilde{P}_\mu &\to \tilde{p}_\mu \ , \\
\mathcal{T}_\nu &\to \sqrt{2}[s^\dagger \tilde{s}]^{01}_{0\nu} \ , \\
\mathcal{S}_\mu &\to \sqrt{2}[p^\dagger \tilde{p}]^{10}_{\mu 0} \ , \\
\mathcal{F}^\mu_\nu &\to -(p^\dagger_\mu \tilde{s}_\nu + s^\dagger_\nu \tilde{p}_\mu) \ \ .
\end{aligned} \tag{37}
$$

Here,

$$
\hat{\mathcal{N}} = -(s^\dagger \cdot \tilde{s} + p^\dagger \cdot \tilde{p})
$$

is the total boson number operator.

As in the SO(5) model, spurious states are introduced by this mapping. Here too they arise for $N > 2\Omega$ [18] and can be readily identified when a boson basis with good spin and isospin is employed.

B. Dynamical symmetries of the model

For certain values of its parameters, the SO(8) Hamiltonian (36) exhibits dynamical symmetries. In such cases, the eigenvalues can be obtained analytically. For all other parameters, analytic expressions for the eigenvalues cannot be obtained, and numerical diagonalization is required.

The $\mathrm{SO^T}(5)$ dynamical symmetry limit of the model is realized when $x = 1$ and $g_{\mathrm{ph}} = 0$. In this case, the Hamiltonian reduces to (1) and its ground-state energy is given analytically by (10).

The $\mathrm{SO^S}(5)$ limit is realized for the parameters $x = -1$ and $g_{\mathrm{ph}} = 0$. In this case, the ground-state energy is given by

$$
E = -g[(\mathcal{N} - T_z)(\Omega + \frac{3}{2} - \frac{1}{2}\mathcal{N} - \frac{1}{2}T_z) - \delta] \ \ . \tag{38}
$$

The third dynamical symmetry arises when $x = 0$, namely when there are equal amounts of isovector and isoscalar pairing. In this case, the Hamiltonian (36) has an SU(4) dynamical symmetry and its exact eigenvalues can be written as

$$
\begin{aligned}
E = &-\frac{1}{4}g[2\mathcal{N}(\Omega+3) - \mathcal{N}^2 - \lambda(\lambda+4)] \\
&+ g_{\mathrm{ph}}[\lambda(\lambda+4) - S(S+1) - T(T+1)] \ \ ,
\end{aligned} \tag{39}
$$

with λ the usual SU(4) label.

The ground-state solution in this symmetry limit arises when $\lambda = T_z + \delta$, and the corresponding energy is

$$
\begin{aligned}
E = &-\frac{1}{4}g[2\mathcal{N}(\Omega+3) - \mathcal{N}^2 - T_z(T_z+4) - \delta(2T_z+5)] \\
&+ 3g_{\mathrm{ph}}(T_z + \delta) \ \ .
\end{aligned} \tag{40}
$$

C. The first boson-mapped Hamiltonian

Mapping the fermion Hamiltonian (36) onto the space of s and p bosons leads to the boson Hamiltonian

$$
\begin{aligned}
H_{\mathrm{B}} = & -\frac{g(1+x)}{2}[(\Omega+1-\hat{\mathcal{N}})s^\dagger\cdot\tilde{s}+\frac{1}{2}(p^\dagger\cdot p^\dagger - s^\dagger\cdot s^\dagger)\tilde{s}\cdot\tilde{s})] \\
& -\frac{g(1-x)}{2}[(\Omega+1-\hat{\mathcal{N}})p^\dagger\cdot\tilde{p}+\frac{1}{2}(s^\dagger\cdot s^\dagger - p^\dagger\cdot p^\dagger)\tilde{p}\cdot\tilde{p})] \\
& +g_{\mathrm{ph}}[p^\dagger\cdot p^\dagger\tilde{s}\cdot\tilde{s}+s^\dagger\cdot s^\dagger\tilde{p}\cdot\tilde{p}+2s^\dagger\cdot\tilde{s}p^\dagger\cdot\tilde{p}+3\hat{\mathcal{N}}] \ .
\end{aligned} \tag{41}
$$

Diagonalization of the boson Hamiltonian (41) is straightforward and represents an alternative method for exactly solving the SO(8) model.

D. Boson mean-field and fermion-pair approximations

It is of interest to consider approximate solutions for the ground state of this model as well. Here, too, the natural approximations to look at first are those based either on mean-field boson methods or on the analogous fermion-pair approximations.

We will not discuss in detail either the methods or the conclusions that emerge from these approximate treatments, since they parallel so closely the discussion for the SO(5) model. Rather, we will just note some of the differences that show up relative to the SO(5) analysis, as derived from the earlier more extensive treatment of the SO(8) model in Ref. [4].

One of the more interesting new features that emerges is a third type of mean-field solution. The solutions A and B, discussed in Sections II and III, of course persist in the SO(8) analysis. But now a third solution C also appears, corresponding to a phase with $\mathrm{n\bar{n}-p\bar{p}}$ and $T=0$ $\mathrm{n\bar{p}-p\bar{n}}$ pairs.

As in the SO(5) analysis, there is a partial decoupling of the pairing phases in the mean-field solutions. Specifically, the $T=1$ $\mathrm{n\bar{p}-p\bar{n}}$ phase is absent from solutions A and C. This leads to significant deficiencies in the mean-field description of $T_z \approx 0$ nuclei, except when isoscalar pairing is dominant.

E. Beyond fermion-pair correlations: A second boson mapping

The boson mean-field and fermion-pair approximations touched on in the previous subsection incorporate fermion-pair correlations. The fact that they are unable to describe the detailed properties of the model suggests the need for a more sophisticated procedure that incorporates further correlations.

1. The T=0 $S=0$ case

The only operators that enter the first boson Hamiltonian (41) are the three operators (24) of the isovector boson space and the three analogous operators

$$p^\dagger \cdot p^\dagger \quad , \quad \widetilde{p} \cdot \widetilde{p} \quad , \quad p^\dagger \cdot \widetilde{p} \tag{42}$$

of the isoscalar boson space. The operators (42) have commutation relations analogous to (25). As a consequence, the Hamiltonian (41) exhibits an O(2,1)⊗O(2,1) symmetry. Furthermore, the two-body terms of this Hamiltonian are precisely in the form of a boson pairing interaction.

As in our treatment of the SO(5) model, we choose to include the effects of boson pairing (or equivalently four-fermion correlations) through the use of a second boson mapping. Furthermore, we follow the same strategy as in Sect. IV, first carrying out the analysis for $T = 0$ $S = 0$ systems, and then generalizing.

When dealing with $T = 0$ $S = 0$ systems, we must map onto a boson space defined by two scalar-isoscalar bosons. One is the t boson introduced in Subsect. IV B, which reflects the correlations of two s bosons. The second, which we denote by q, reflects the correlations of two p bosons. The Dyson realization of the O(2,1)⊗O(2,1) algebra contains two parts. The set of operators (24) of the isovector boson space map according to (26). The set of operators in the isoscalar space map according to the analogous relations

$$\begin{aligned} p^\dagger \cdot p^\dagger &\to 6q^\dagger + 4q^\dagger q^\dagger q \ , \\ \widetilde{p} \cdot \widetilde{p} &\to q \ , \\ p^\dagger \cdot \widetilde{p} &\to -2q^\dagger q \ . \end{aligned} \tag{43}$$

Applying the second boson mapping to the Hamiltonian (41) leads to a U(1)⊗U(1) (or SU(2)) Hamiltonian

$$\begin{aligned} H_{\rm BB} = & -g(1+x)[(\Omega + \frac{5}{2} - \hat{\mathcal{N}})t^\dagger t + t^\dagger t^\dagger t t - \frac{3}{2}q^\dagger t - q^\dagger q^\dagger q t] \\ & -g(1-x)[(\Omega + \frac{5}{2} - \hat{\mathcal{N}})q^\dagger q + q^\dagger q^\dagger q q - \frac{3}{2}t^\dagger q - t^\dagger t^\dagger t q \ ] \\ & +g_{\rm ph}[6(q^\dagger t + t^\dagger q) + 4(q^\dagger q^\dagger q t + t^\dagger t^\dagger t q) + 8t^\dagger t q^\dagger q + 3\hat{\mathcal{N}}] \ , \end{aligned} \tag{44}$$

where now $\hat{\mathcal{N}}$ is given by

$$\hat{\mathcal{N}} = 2(t^\dagger t + q^\dagger q) \ .$$

In general, the problem of the Hamiltonian (44) must be solved by numerical matrix diagonalization. In the case of the dynamical symmetry limits, however, we can obtain simple analytic solutions.

In the SO$^{\rm T}$(5) limit, the Hamiltonian matrix in the $|t^{\frac{1}{2}\mathcal{N}-n_q}q^{n_q}))$ basis has all zeros below the diagonal. The same considerations as in Subsect. IV B 2 can therefore be applied. The diagonal elements of the Hamiltonian matrix give directly the eigenvalues and reproduce the exact result (9) for the $T = 0$ energy when an identification $n_q = \frac{1}{4}v_{\rm s}$ is made. The same is true for the SO$^{\rm S}$(5) limit, with the roles of the isospin and spin and also the t- and q-bosons interchanged.

To obtain the solution in the SU(4) limit, we first impose the transformation

$$\begin{aligned} r^\dagger &= \frac{1}{\sqrt{2}}(t^\dagger - q^\dagger) \ , \\ w^\dagger &= \frac{1}{\sqrt{2}}(t^\dagger + q^\dagger) \ , \end{aligned} \tag{45}$$

and rewrite the Hamiltonian (44) as

$$\begin{aligned} H_{\rm BB} = -g[\frac{1}{2}(\Omega + \frac{5}{2} - \hat{\mathcal{N}})\hat{\mathcal{N}} + \frac{3}{2}(r^\dagger r - w^\dagger w) \\ + r^\dagger r^\dagger r r + 2 r^\dagger r w^\dagger w + w^\dagger w^\dagger r r] \\ + 4 g_{\rm ph}[3 w^\dagger w + w^\dagger w^\dagger w w - w^\dagger w^\dagger r r] \ . \end{aligned} \tag{46}$$

Working with this Hamiltonian in the basis $|r^{\frac{1}{2}\mathcal{N}-n_w} w^{n_w}))$, we recover the exact SU(4) eigenvalues (39), when an identification $n_w = \frac{1}{2}\lambda$ is adopted.

2. General case

The procedure just outlined for carrying out a second boson mapping to the $t-\sigma$ space enables investigations of $\mathcal{N}=$ even $T=0$ $S=0$ states only. For a more complete treatment, we must appropriately extend the second boson mapping, in much the same way as we did in Subsection IV B 2. Namely, we must include, in addition to the sets of operators (24), (29) and (42), the additional one-boson creation and annihilation operators

$$p^\dagger \quad , \quad \widetilde{p} \ . \tag{47}$$

The full second boson mapping is now given by (31) and

$$\begin{aligned} p^\dagger \cdot p^\dagger &\to 6 q^\dagger + 4 q^\dagger q^\dagger q + \pi^\dagger \cdot \pi^\dagger - 4 q^\dagger \pi^\dagger \cdot \widetilde{\pi} \ , \\ \widetilde{p} \cdot \widetilde{p} &\to q \ , \\ p^\dagger \cdot \widetilde{p} &\to -2 q^\dagger q + \pi^\dagger \cdot \widetilde{\pi} \ , \\ p^\dagger_\nu &\to \pi^\dagger_\nu - 2 q^\dagger \widetilde{\pi}_\nu \ , \\ \widetilde{p}_\nu &\to \widetilde{\pi}_\nu \ , \end{aligned} \tag{48}$$

where π denotes the additional vector-isoscalar boson needed to complete the ideal boson space.

This ideal boson space – formed from the t, σ, q, and π bosons – contains an unphysical subspace. Since the sectors t-σ and q-π are separated and since we have already determined the physical states (33) in the t-σ sector, we find that the physical basis of the full problem is of the form

$$|t^{\frac{1}{2}(\mathcal{N}-T-S)-n_q} q^{n_q} \ ; \quad \sigma^T T ; \pi^S S)) \quad , \quad S + T \leq 2\Omega - \mathcal{N} \ . \tag{49}$$

The second boson image of the SO(8) Hamiltonian is straightforwardly obtained and in general can be solved by matrix diagonalization. We will discuss here only the dynamical symmetry limits, where analytical solutions can be obtained as in Subsection IV B 2.

In the $\mathrm{SO^T}(5)$ limit, for example, the Hamiltonian takes the form

$$\begin{aligned} H_{\rm BB} = -g[(\Omega + 1 - \hat{\mathcal{N}})(2 t^\dagger t - \sigma^\dagger \cdot \widetilde{\sigma}) \\ + 3 t^\dagger t + 2 t^\dagger t^\dagger t t - 2 t^\dagger t \sigma^\dagger \cdot \widetilde{\sigma} + \frac{1}{2}\sigma^\dagger \cdot \sigma^\dagger t \\ - 3 q^\dagger t - 2 q^\dagger q^\dagger q t + 2 q^\dagger t \pi^\dagger \cdot \widetilde{\pi} - \frac{1}{2}\pi^\dagger \cdot \pi^\dagger t] \ , \end{aligned} \tag{50}$$

with

$$\hat{\mathcal{N}} = 2(t^\dagger t + q^\dagger q) - \sigma^\dagger \cdot \widetilde{\sigma} - \pi^\dagger \cdot \widetilde{\pi} \quad .$$

In the basis (49), the exact energies (9) are obtained when the identification $2n_q + S = \frac{1}{2}v_s$ is made. The left vectors of the basis (49) give the left physical eigenvectors in this limit.

The same arguments can be applied to the $\mathrm{SO^S}(5)$ limit. The only difference is that we must interchange the role of the isospin with that of the spin, and likewise the role of the t- and σ- bosons with that of the q- and π-bosons.

In the SU(4) limit, we again apply the transformation (45), after which the second boson Hamiltonian takes the form

$$\begin{aligned} H_{\mathrm{BB}} = & -g[\frac{1}{2}(\Omega + 1 - \hat{\mathcal{N}})\hat{\mathcal{N}} + 3r^\dagger r + r^\dagger r^\dagger rr \\ & +2r^\dagger r w^\dagger w + w^\dagger w^\dagger rr - r^\dagger r(\sigma^\dagger \cdot \widetilde{\sigma} + \pi^\dagger \cdot \widetilde{\pi}) \\ & -w^\dagger r(\sigma^\dagger \cdot \widetilde{\sigma} - \pi^\dagger \cdot \widetilde{\pi}) + \frac{\sqrt{2}}{4}(\sigma^\dagger \cdot \sigma^\dagger - \pi^\dagger \cdot \pi^\dagger)r] \\ & +g_{\mathrm{ph}}[6w^\dagger w - 6r^\dagger r + 4w^\dagger w^\dagger ww - 4w^\dagger w^\dagger rr \\ & +3\hat{\mathcal{N}} - 4w^\dagger w(\sigma^\dagger \cdot \widetilde{\sigma} + \pi^\dagger \cdot \widetilde{\pi}) + 4w^\dagger r(\sigma^\dagger \cdot \widetilde{\sigma} - \pi^\dagger \cdot \widetilde{\pi}) \\ & +\frac{1}{\sqrt{2}}(\sigma^\dagger \cdot \sigma^\dagger + \pi^\dagger \cdot \pi^\dagger)w - \frac{1}{\sqrt{2}}(\sigma^\dagger \cdot \sigma^\dagger - \pi^\dagger \cdot \pi^\dagger)r + 2\sigma^\dagger \cdot \widetilde{\sigma}\pi^\dagger \cdot \widetilde{\pi}] \quad . \end{aligned} \tag{51}$$

Again, the SU(4) energies (39) are easily obtained by working in the basis

$$|r^{\frac{1}{2}(\mathcal{N}-T-S)-n_w} w^{n_w};\ \sigma^T T\ ;\ \pi^S S)) \ , \tag{52}$$

with $2n_w + S + T = \lambda$. The left vectors of the basis (52) give the left physical eigenvectors in the SU(4) limit.

F. Wave function in terms of fermion-pair operators

As we have just seen, the left physical eigenvectors of the second boson-mapped space take a very simple form in the three dynamical symmetry limits. As in Subsection IV C, we can use this fact to find the corresponding wave functions in terms of fermion-pair operators. To do this, we must use the chains (34) as well as the analogous P-sector chains

$$\begin{aligned} \widetilde{P} \cdot \widetilde{P} &\rightarrow \widetilde{p} \cdot \widetilde{p} \rightarrow q \ , \\ \widetilde{P}_\mu &\rightarrow \widetilde{p}_\mu \rightarrow \widetilde{\pi}_\mu \ . \end{aligned} \tag{53}$$

.

Proceeding as in Subsect. IV C, we find that in the $\mathrm{SO^T}(5)$ limit the wave functions are given by

$$\begin{aligned} |\mathcal{N}TT_zSS_zv_s\rangle \propto & \mathcal{P}_{T_z}\mathcal{P}_{S_z}(S^\dagger \cdot S^\dagger)^{\frac{1}{2}(\mathcal{N}-T-\frac{1}{2}v_s)}(P^\dagger \cdot P^\dagger)^{\frac{1}{2}(\frac{1}{2}v_s - S)} \\ & \times(\frac{1}{2}S_1^\dagger + \frac{1}{2}S_{-1}^\dagger + \frac{1}{\sqrt{2}}S_0^\dagger)^T(\frac{1}{2}P_1^\dagger + \frac{1}{2}P_{-1}^\dagger + \frac{1}{\sqrt{2}}P_0^\dagger)^S|0\rangle \ , \end{aligned} \tag{54}$$

where $\mathcal{P}_{S_z}$ is the projection operator that picks states with spin-projection S_z.

An analogous expression holds in the $\mathrm{SO^S}(5)$ limit with the roles of isospin and spin and also of $S^\dagger$ and $P^\dagger$ interchanged.

In the SU(4) limit, we must again take into account the transformation (45). This then leads to

$$|\mathcal{N}TT_zSS_z\lambda\rangle \propto \mathcal{P}_{T_z}\mathcal{P}_{S_z}(S^\dagger\cdot S^\dagger - P^\dagger\cdot P^\dagger)^{\frac{1}{2}(\mathcal{N}-\lambda)}(S^\dagger\cdot S^\dagger + P^\dagger\cdot P^\dagger)^{\frac{1}{2}(\lambda-S-T)} \\ \times(\frac{1}{2}S_1^\dagger + \frac{1}{2}S_{-1}^\dagger + \frac{1}{\sqrt{2}}S_0^\dagger)^T(\frac{1}{2}P_1^\dagger + \frac{1}{2}P_{-1}^\dagger + \frac{1}{\sqrt{2}}P_0^\dagger)^S|0\rangle \ . \tag{55}$$

In the case of the ground-state solutions, the wave functions associated with the three dynamical symmetry limits can be unified. For even-even $N = Z$ nuclei, for example, all three can be written as

$$|\mathcal{N}\,T\!=\!0\,T_z\!=\!0\,S\!=\!0\rangle \propto (\alpha S^\dagger\cdot S^\dagger - \beta P^\dagger\cdot P^\dagger)^{\mathcal{N}/2}|0\rangle \ . \tag{56}$$

In the $\mathrm{SO^T}(5)$ limit, $\alpha = 1$ and $\beta = 0$. In the $\mathrm{SO^S}(5)$ limit, $\alpha = 0$ and $\beta = 1$. And, in the SU(4) limit, $\alpha = \beta = 1/\sqrt{2}$. [14]

Similarly, we can write the ground-state wave function for even-even $N > Z$ nuclei by adding the appropriate number of isovector nn pairs to the $N = Z$ solution (56), viz:

$$|\mathcal{N}\,T\!=\!T_z\,T_z\,S\!=\!0\rangle \\ \propto (\alpha S^\dagger\cdot S^\dagger - \beta P^\dagger\cdot P^\dagger)^{(\mathcal{N}-T_z)/2}S_1^{\dagger T_z}|0\rangle \ . \tag{57}$$

Finally, for odd-odd nuclei, we must add to (57) either an isoscalar or an isovector np pair, depending on the symmetry limit. In particular, in the $\mathrm{SO^T}(5)$ and SU(4) limits, the ground state is given by

$$|\mathcal{N}\,T\!=\!T_z\!+\!1\,T_z\,S\!=\!0\rangle \\ \propto (\alpha S^\dagger\cdot S^\dagger - \beta P^\dagger\cdot P^\dagger)^{(\mathcal{N}-T_z-1)/2}S_1^{\dagger T_z}S_0^\dagger|0\rangle \ . \tag{58}$$

Analogously, in the $\mathrm{SO^S}(5)$ and SU(4) limits, it is given by[15]

$$|\mathcal{N}\,T\!=\!T_z\,T_z\,S\!=\!1\rangle \\ \propto (\alpha S^\dagger\cdot S^\dagger - \beta P^\dagger\cdot P^\dagger)^{(\mathcal{N}-T_z-1)/2}S_1^{\dagger T_z}P^\dagger|0\rangle \ . \tag{59}$$

In the SO(8) model, the quartet creation operator that represents an alpha cluster takes the form [11][16]

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[14]Here, the normalization $\alpha^2 + \beta^2 = 1$ is used.

[15]In the SU(4) limit, there is a degeneracy in the ground states of odd-odd nuclei, explaining why we give two distinct ground-state solutions.

[16]There is a sign mistake in Ref. [11]

$$\frac{1}{2\sqrt{3\Omega(\Omega+2)}}(S^\dagger \cdot S^\dagger - P^\dagger \cdot P^\dagger)\ .$$

We see, therefore, that in the SU(4) limit the ground state involves as many alpha-correlated structures as possible.

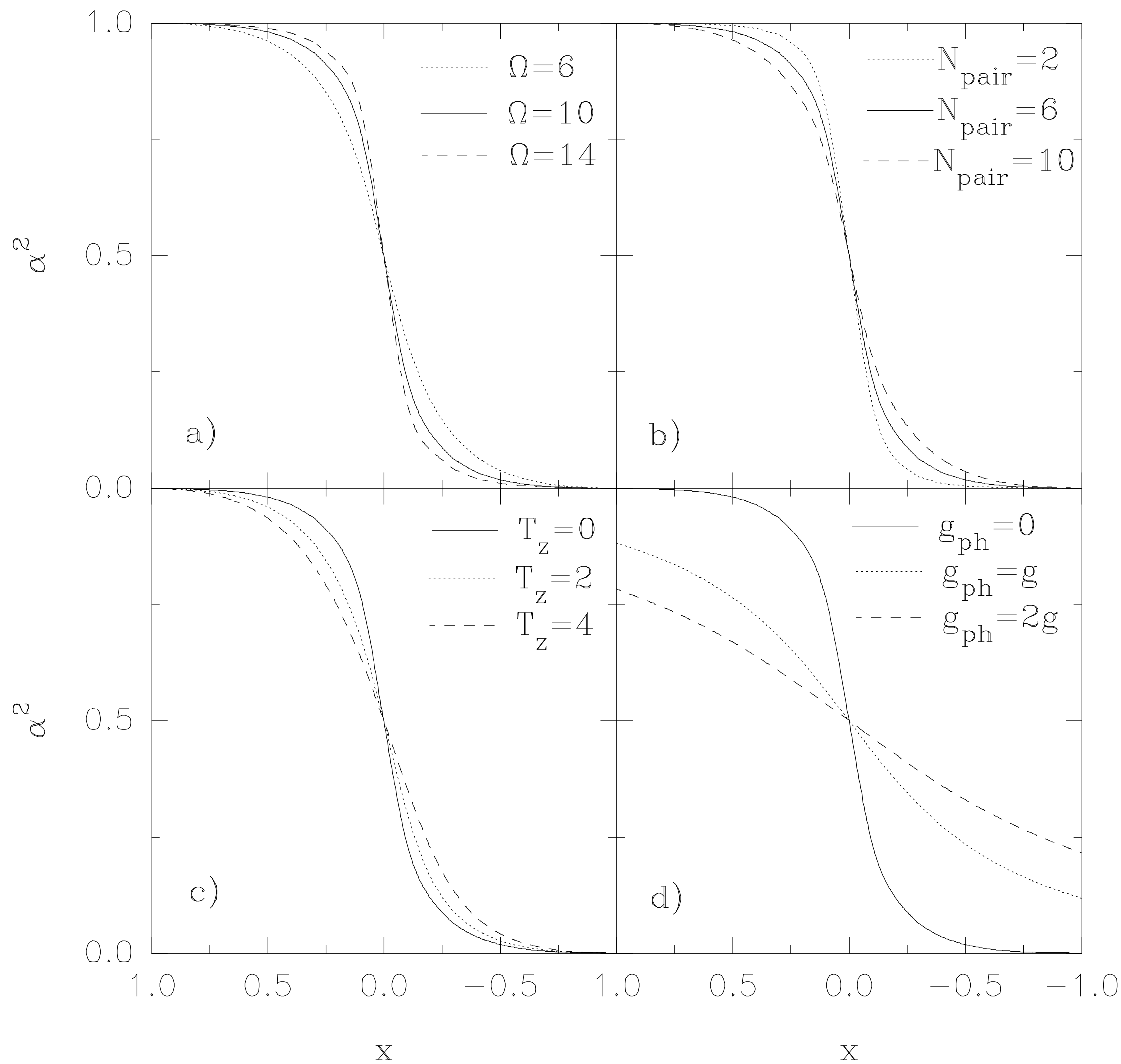

FIG. 3. The variational parameter α^2 in the approximate SO(8) ground-state wave functions (54-57) as a function of the Hamiltonian parameter x that controls the balance between isoscalar and isovector pairing. The behavior is shown for a) various shell degeneracies Ω with fixed $\mathcal{N}=6$ and $T_z=0$, b) various $N_{\rm pair}=\mathcal{N}$ with $\Omega=10$ and $T_z=0$, c) various T_z with fixed $\Omega=10$ and $\mathcal{N}=T_z+6$, and d) various values of the particle-hole strength $g_{\rm ph}$ in the $T=1, S=1$ channel with fixed $\mathcal{N}=6$, $T_z=0$, and $\Omega=10$.

Outside the dynamical symmetry limits, the above forms for the ground-state wave function are not exact. Nevertheless, when we consider them as variational wave functions (with α and β as variational parameters), we find that at the minimum they have almost perfect overlap with the exact wave functions. Fig. 3 illustrates the behavior of α^2 as a

function of the parameter x of the SO(8) Hamiltonian for a few representative cases.[17]

We first discuss the results in the limit $g_{\rm ph} = 0$, as illustrated in Fig. 3a-c. In this limit, we are in the pure isovector phase at $x = 1$ and in the pure isoscalar phase at $x = -1$. Furthermore, the transition between the two phases becomes sharper as Ω increases and/or the number of pairs decreases. In passing from the one phase to the other, we of course pass through the SU(4) phase. Note, however, that this is done smoothly. There is no plateau at the SU(4) phase, suggesting no special stability associated with the maximally alpha-correlated state.

When a particle-hole force is turned on [see Fig. 3d], some interesting changes show up. In particular, the pure isoscalar and isovector phases are never realized. And, furthermore, the transition through the SU(4) phase is less abrupt. This latter remark may have relevance to real nuclear systems, which are most likely near to – but not precisely at – the SU(4) limit. Our results suggest that even away from the SU(4) limit, there may be significant alpha-particle transfer strength, as long as there is a sufficiently strong particle-hole force present.

Finally, we close this section by reiterating the key conclusion of our analysis of the SO(8) model. Namely, in the SO(8) model – a prototypical model that involves np pairing correlations on the same footing as nn and pp pairing correlations – the ground-state wave functions exhibit essential four-particle correlations. The fermion SO(8) ground state is constructed in such a way that the maximal possible number of fermions form correlated four-particle $S = 0$ $T = 0$ structures and the rest form like-particle pairs and/or an np pair and provide the isospin and spin of the system.

VI. DISCUSSION

We have studied neutron-proton pairing correlations within the framework of two simple, solvable, and somewhat realistic nuclear models. In our analysis, we made extensive use of boson-mapping techniques. Since the bosons introduced by the (first) mapping represent correlated fermion pairs, such an approach provides a quite convenient framework in which to investigate pairing features in fermion systems.

With that in mind, we first studied the isovector-pairing SO(5) model, both within the framework of boson mean-field methods and also through the use of genuine fermion-pair approximations. The latter represent generalizations of the standard procedures to treat the pairing between like nucleons. We find that these approaches fail to describe details of neutron-proton pairing. In particular, the BCS solution does not allow for the coexistence of like-particle and neutron-proton pairs. As a result, two-nucleon transfer strengths are not given correctly, even though the approximate energies are quite close to the exact values. Tiny details of neutron-proton pairing correlations, as reflected in double binding energy differences, are also not well reproduced. Number and T_z projection, by themselves, are unable to improve the situation.

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[17]In the odd-odd case, the form (58) applies for $x \geq 0$ and the form (59) for $x \leq 0$.

A nice feature of the boson mapping procedure is that it provides detailed information on the boson-mapped Hamiltonian, information that can yield useful clues as to how to introduce the necessary additional correlations. The boson Hamiltonian that emerged from a mapping of the SO(5) fermion Hamiltonian contained an attractive pairing force between bosons, suggesting that correlations between pairs of bosons could be the key missing ingredient to an improved description of the system. After first exploring the possible use of boson BCS approximation to accomodate these additional correlations, we then turned to an alternative treatment based on a second boson mapping. A simple description of the neutron-proton pairing problem was achieved in terms of the second bosons, confirming the importance of boson-boson (or equivalently pair-pair) correlations. Inverting the method, we were also able to obtain in closed form the fermion wave functions of the SO(5) model expressed in terms of the fundamental fermion-pair operators. The wave functions that emerged clearly showed four-nucleon correlations.

We then carried out a similar study of a somewhat richer model involving neutron-proton pair correlations based on the algebra SO(8). We focussed our analysis on the dynamical symmetry limits of this model, where the group structure could be used to obtain simple closed expressions. Here too we found that a description limited to fermion-pair (or boson mean-field) correlations was insufficient and that four-particle correlations were needed to accurately reproduce the exact results.

Thus, from these model calculations, we conclude that correlations involving pairs of fermion pairs, or alternatively quartets of fermions, are important in the regime of neutron-proton pairing. It is not sufficient only to pair nucleons. Whenever possible, two nucleon pairs will couple together to form a $T = 0$ $S = 0$ (alpha-particle-like) structure.

Qualitatively, this conclusion can be understood as follows. The smallest "cluster" that can simultaneously accomodate two-neutron pairing correlations, two-proton pairing correlations and neutron-proton pairing correlations is one that involves four nucleons – two neutrons and two protons. Of course, when there is an excess of particles of a given type, not all particles can form these maximally-correlated alpha-like structures. Instead, they remain in like-particle and/or neutron-proton pairs, appended to the alpha-like *condensate*.

The above conclusion is not limited, however, to cases in which all pairing modes contribute significantly. In the $\mathrm{SO^S}(5)$ limit of the SO(8) model, for example, which only involves isoscalar np pairing, the ground-state wave function involves a condensate of $P^\dagger \cdot P^\dagger$ pairs and thus contains four-particle correlations. There, however, the fact that the ground state involves such a quartet structure is a direct reflection of angular momentum restoration.

It is important to note, however, that the four-particle correlated structures that emerge are not exact alpha particles. Nor are they necessarily the most alpha-like structures available within the model. In the SO(8) model, for example, the ground state is dominated by the maximal alpha structure in the SU(4) dynamical symmetry limit only.

A challenging problem that still remains is: What is a good way to incorporate such four-fermion correlations into nuclear many-body approximation schemes? As noted above, boson BCS following a boson mapping does not seem to be an acceptable procedure for incorporating four-nucleon correlations in such models; fluctuations in the particle number are simply too large. Another possible approach would be to include the additional correlations through projection. Isospin projection, in particular, would seem to be important. In the simple SO(5) model, for example, it leads to the exact solution. It still remains, however,

to develop an appropriate isospin projection technique and to apply it to realistic nuclear structure problems.

We have put forth two ideas that could perhaps provide the basis for an improved theory with four-particle correlations. One possibility would be to start from a trial ground-state-wave function in the form of Eqs. (56)-(59). The $S^\dagger$ and $P^\dagger$ are now collective pair creation operators, whose structure ideally should be determined variationally (or, less ideally, from an analysis of simple two-body systems). Such an approach is a generalization of the generalized-seniority scheme for like nucleons, which is known to be connected to number-projected BCS theory.

A second possibility worth further investigation is the use of iterative boson mappings. We have seen that two boson mappings, coupled with a Hartree-Bose treatment in the second boson space, is a way to build a number-projected theory involving four-nucleon correlations. In the current studies, where spurious states could be readily identified, such an approach proved extremely useful. Whether it will continue to prove useful in more realistic applications, however, where spurious states cannot be so readily separated, still remains to be investigated.

Other truncation schemes in the shell model should also be studied. Our analysis of the SO(8) model suggests that the usual truncation schemes built in terms of separate dynamical symmetry limits for neutrons and protons (the SU(2) seniority limit, for example) should not be of much use in the general neutron-proton pairing problem. A truncation in the SO(5) or SO(8) seniority quantum numbers, which would then include the necessary isospin correlations, could perhaps be useful.

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